Periodic D-Pullback Attractor of Strongly Damped Sine-Gordon Equation with Memory

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Abstract: We study the existence of periodic pullback attractor for non-autonomous dynamical system generated by Sine-Gordon equation with non-autonomous term as well as memory. By a priori estimation, we obtain the global solution of the equation and the pullback absorting set of ; by constructing the contraction function, we prove the asymptotic compactness of , then the existence of pullback attractor of the corresponding non-autonomous dynamical system in is established; moreover the pullback attractor is proved to be periodic, when the non-autonomous dynamical system has a periodic deterministic forcing term.

1. Introduction

In this paper, we study the asymptotic behavior of Nonautonomous sine Gordon equation with memory:

$$\begin{cases} u_{tt}-\Delta u_t+\Delta^2 u+\int_0^{\infty} g(s)\Delta\big(a(x)\Delta u_t(t-s)\big)\mathrm{d}s+f(\sin u)=h(x,t), \\ u=0,\ \Delta u=0,\ x\in\Gamma, t\geq\tau, \\ u(x,\tau)=u_0(x), u_t(x,\tau)=\partial_t u_0(x), (x,t)\in\Omega\times[\tau,\infty). \end{cases} \tag{1}$$

Here $\Omega\subset R^N$ is a bounded domain with smooth bounded Γ, $u_0:\Omega\times(-\infty,0]\to R$ is the Past history of u.

It is an important problem in mathematical physics to study the long-time dynamic behavior of the solution to problem (1) which is a dissipative infinite dimensional dynamic system. When the external force term h is autonomous, there are many important research results on the long-term dynamic behavior of the solution. For details, please refer to the literature [1-4].Zhang Jianwen, Ren Yonghua, Wu runheng and Feng Tao[5] The dynamic behavior of nonlinear thermoelastic coupled sine-Gordon type system under the action of Peierls Nabarro force is studied. In this paper, the continuous solution of the system under certain initial boundary conditions is obtained by using the operator semigroup theory, and the asymptotically compact invariant absorption set is constructed by using the decomposition method of operator semigroup. Finally, the existence of global attractor is obtained. When the external force term h is nonautonomous, papers mainly studies its pull back characteristics, The pull back attractors of various equations are studied in the literatures[6-9].

Among the nonlinear evolution equations, the study of sine Gordon equation has been favored by many scholars,Under the condition of autonomy, The global solution[10],global attractor[11] and dimension estimation[11, 12]is studied in articles[10-12];Under the condition of nonautonomy, The study of sine-Gordon type equation is mainly about pullback dynamics[13-15],The pullback dynamics of Nonautonomous stochastic sine Gordon equations with additive perturbation is studied in[13], The existence of a pullback attractor on $\left(H_0^1(O)\times L^2(O)\right)^2$ for the stochastic dynamical system corresponding to the system of equations is proved by using the method of uniform estimation of solutions. More results of pullback dynamics can be found in [16-21].

Some scholars have studied the asymptotic behavior of differential equations with linear memory. See [3-4,6-7]. However, the asymptotic behavior of the generalized nonautonomous sine-Gordon type equations with damping memory has not been studied. Combined with the research results of related scholars,The D-pullback attractor of Nonautonomous sine Gordon equation with memory term is studied. The D-pullback asymptotically compactness of continuous cocycle Φ of problem (1) is obtained by the method of contraction function, and then the D-pullback attractor of problem (1) is obtained. At the same time, it is discussed that when the non autonomous external force term has periodicity about t, its corresponding D-pullback attractor also has the same periodicity.

In order to make the research smooth, some hypotheses are given

(E_1) $f \in C^2(R)$,And meet the following conditions $f(s) \le C_0\left(1+|s|^q\right)$,here $q>0$;

(E_2)Memory kernel function $\forall s \in R^+,\ g(\bullet)\in C^2\left(R^+\right),\ g'(s) \le 0 \le g(s),\ g(+\infty)=0$, $\mu(s)=-g'(s)$,And satisfied

(I) $\mu \in C^1\left(R^+\right)\cap L^1\left(R^+\right),\ \mu'(s)\le 0 \le \mu(s),\ \forall s \in R^+$,

(II) $\mu_0=\int_0^\infty \mu(s)\mathrm{d}s>0,\ \mu'(s)+\mu_1\mu(s)\le 0,\ \forall s\in R^+$, μ_1 is a positive constant

The problem (1) is transformed into a definite nonautonomous dynamical system, Therefore, combined with the method of reference [3, 7] , the historical displacement variable is introduced, namely

$\eta=\eta^t(x,s)=\int_0^s u_t(x,t-r)\mathrm{d}r,\ (x,s)\in\Omega\times R^+,\ t\ge 0$, (2)

By formal differentiation we have

$\eta_t^t(x,s)=-\eta_s^t(x,s)+u_t(x,t),\ (x,s)\in\Omega\times R^+,\ t\ge 0$. (3)

according to (E_2) $\mu(s)=-g'(s)$,and $g(+\infty)=0$,Then problem (1) can be transformed into the following equivalent equations:

$$\begin{cases} u_{tt}-\Delta u_t+\Delta^2 u-\int_0^\infty \mu(s)\Delta\left(a(x)\Delta\eta^t(s)\right)\mathrm{d}s+f(\sin u)=h(x,t), & (x,t)\in\Omega\times[\tau,\infty), \\ \eta_t=-\eta_s+u_t,\ (x,t,s)\in\Omega\times R^+\times R^+. \end{cases} \tag{4}$$

Initial condition:

$u(x,\tau)=u_0(x),\ u_t(x,\tau)=u_1(x),\ \eta^\tau(x,s)=\eta_0(x,s),\ \eta^t(x,0)=0$。 (5)

Corresponding boundary conditions:

$u=\Delta u=0,\ (x,t)\in\Gamma\times R^+,\ \eta=\Delta\eta=0,\ x\in\Gamma,\ t\ge\tau$。 (6)

2. Preliminaries

Without losing its generality, Let $(\bullet,\bullet)$ and $\|\bullet\|$ denote the inner product and norm of $L^2(\Omega)$, respectively

$H=V_0=L^2(\Omega),V_2=H^2(\Omega)\cap H_0^1(\Omega)$,

The corresponding inner product and norm are

$(u,v)_{V_2}=(\Delta u,\Delta v),\|u\|_{V_2}=\|\Delta u\|_H$。

Let $a(x)\in C^1\left(\overline{\Omega}\right)$,then

$meas\left\{x\in\Gamma\,\middle|\,a(x)>0\right\}>0$,

and

$V_a=\left\{u\in H\,\middle|\,\int_\Omega a(x)|\Delta u|^2\mathrm{d}x<\infty,u|_\Gamma=\Delta u|_\Gamma=0\right\}$, Easy to know, V_a is a Hilbert space whose norm is:

$\|u\|_{V_a}^2 = \int_\Omega a(x)|\Delta u|^2 \,\mathrm{d}x$。

According to the above definition, the history space is given

$$\mathrm{M}=L_\mu^2\left(R^+;V_a\right)=\left\{\eta : R^+ \to V_a \left| \int_0^\infty \mu(s)\|\eta(s)\|_{V_a}^2 \,\mathrm{d}s < \infty \right.\right\},$$

The history space is also a Hilbert space, and the corresponding inner product and norm:

$$(\eta,\xi)_{\mathrm{M}} = \int_0^\infty \mu(s)\int_\Omega a(x)\Delta\eta(x,s)\bullet\Delta\xi(x,s)\,\mathrm{d}x\mathrm{d}s,\ \|\eta\|_{\mathrm{M}}^2 = \int_0^\infty \mu(s)\|\eta(s)\|_{V_a}^2 \,\mathrm{d}s.$$

According to Poincare inequality: $\lambda_1\left\|\nabla^r u\right\|^2 \le \left\|\nabla^{r+1}u\right\|^2$,here λ_1 is the first eigenvalue of $-\Delta$.

In order to get our results, the related concepts and theories are given.

Definition1.1[18] Let mapping $\theta_t : \Omega_1 \to \Omega_1, t \in R$. $\left(\Omega_1,\{\theta_t\}_{t\in R}\right)$ is called parametric dynamical system, if the following conditions are met:

1) θ_0 is an identity map defined on Ω_1 ,i.e $\forall q \in \Omega_1$, $\theta_0(q) = q$;

2) $\theta_{t+\tau}(q) = \theta_t(\theta_\tau(q))$;

3) $(t,q) \to \theta_t(q)$ is continuous.

Definition 1.2[19] $\left(\Omega_1,\{\theta_t\}_{t\in R}\right)$ is parametric dynamical system,mapping Φ: $R^+ \times \Omega_1 \times X \to X$ is called a continuous cocycle dominated by $\left(\Omega_1,\{\theta_t\}_{t\in R}\right)$ on X ,if for $\forall q \in \Omega_1$ 和 $t,\tau \in R^+$,satisfying:

1) $\Phi(0,q,\bullet)$ is an identity map defined on X ,i.e $\forall (q,x) \in \Omega_1 \times X$, $\Phi(0,q,x) = x$;

2) $\Phi(t+\tau,q,x) = \Phi(\tau,\theta_t(q),\Phi(t,q,x))$;

3) $\Phi(t,q,\bullet): X \to X$ is continuous.

If, in addition, there exists a positive number T_0 ,such that for every $t \ge 0, q \in \Omega_1$, $\Phi\left(t,\theta_{T_0}(q),\bullet\right) = \Phi(t,q,\bullet)$,then Φ is called a continuous periodic cocycle on X with period T_0.

Let $\mathrm{P}(X)$ denote the family of all nonempty subsets of X ,and S is the class of all families $\hat{D} = \{D(q) : q \in \Omega_1\} \subset \mathrm{P}(X)$.Then we consider given a nonempty subclass $\mathrm{D} \subset S$.

Definition 1.3 Let T_0 be a positive number,and $\hat{D} = \{D(q) : q \in \Omega_1\} \subset \mathrm{P}(X)$, all $q \in \Omega_1$, $D\left(\theta_{T_0}(q)\right) = D(q)$, Then $\hat{D}$ is said to have period T_0.

Definition 1.4(contractive function)[20] Let X is a banach space, B is a bounded subset of X, $\Psi(\bullet,\bullet)$ is the function defined on $X \times X$ .if for any sequence $\{x_n\}_{n=1}^\infty \subset B$,there is a subsequence $\{x_{nk}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$,such that

$\lim\limits_{k\to\infty}\lim\limits_{l\to\infty}\Psi\left(x_{nk},x_{nl}\right) = 0$. (7)

We call $\Psi(\bullet,\bullet)$ a contractive function on $B \times B$

We denote the set of all contractive functions on $B \times B$ by $contr(B)$.

Definition 1.5(D- pullback absorbing set)[19, 20] if for any $q \in \Omega_1$, $\hat{D} \in \mathrm{D}$,there exists

$t_0 = t_0\left(q,\hat{D}\right) \ge 0$

such that

$\Phi\left(t,\theta_{-t}(q),D\left(\theta_{-t}(q)\right)\right) \subset B(q)$,对 $\forall t \ge t_0$,

A family of bounded sets $\hat{B} = \{B(q) : q \in \Omega_1\}$ is D- Pullback absorbing.

Definition 1.6(D- pullback attractor)[19] A family of nonempty compact sets $\hat{A}=\{A(q):q\in\Omega_1\}\in S$ is a D- pullback attractor,if it satisfies:

1) for any $\forall q\in\Omega_1$, $A(q)$ is compact;

2) $\hat{A}$ is D- pullback attracting,i.e for all $\hat{D}\in \mathrm{D}$, $q\in\Omega_1$,

$\lim_{t\to+\infty} dist\Big(\Phi\big(t,\theta_{-t}(q),D(\theta_{-t}(q))\big),A(q)\Big)=0$,

Here $dist(Q_1,Q_2)=\sup_{x\in Q_1}\inf_{y\in Q_2} d(x,y)$, $Q_1,Q_2\subset X$, $d(\bullet,\bullet)$ is the distance of space;

3) $\hat{A}$ is invariant,i.e for all $(t,q)\in R^+\times\Omega_1$,

$\Phi(t,q,A(q))=A(\theta_t(q))$.

there exists $T_0>0$,such that $A\big(\theta_{T_0}(q)\big)=A(q)$, $\forall q\in\Omega_1$,then we say $\hat{A}$ is periodic with period T_0.

Definition 1.7[19] For each $\hat{D}\in S$, $q\in\Omega_1$,we define the omega-limit set of $\hat{D}$ at q (in the pullback sense)as $\Lambda(\hat{D},q)=\bigcap_{s\ge0}\overline{\bigcup_{t\ge s}\Phi\big(t,\theta_{-t}(q),D(\theta_{-t}(q))\big)}$.

Theorem1.1[19] Suppose the continuous cocycle Φ is D- pullback asymptotically compact,and there exists $\hat{B}\in\mathrm{D}$ which is D- pullback absorbing. Then, the family $\hat{A}$ defined by $A(q)=\Lambda(\hat{B},q)(q\in\Omega_1)$ is a D- pullback attractor.

Theorem1.2[20] Let Φ is a continuous cocycle in $\big(\Omega_1,\{\theta_t\}_{t\in R}\big)$,suppose Φ has a D- pullback absorbing set $\hat{B}=\{B_q\}_{q\in\Omega_1}$,moreover for any $\nu>0,q\in\Omega_1$,there exists $T=T(q,\nu)$ and $\Psi_{T,q}(\bullet,\bullet)\in contr\big(B_{\theta_{-T}(q)}\big)$,such that

$\big\|\Phi(t,\theta_{-T}(q),x)-\Phi(t,\theta_{-T}(q),y)\big\|_X\le\nu+\Psi_{T,q}(x,y)$,for $\forall x,y\in B_{\theta_{-T}(q)}$

where $\Psi_{T,q}$ depends on T and q.

Then Φ is D- pullback asymptotically in X .

Theorem1.3[14, 21] Suppose Φ be a continuous periodic cocycle with period $T_0>0$ on X over $\big(\Omega_1,\{\theta_t\}_{t\in R}\big)$,if Φ has a D- puback attractor $\hat{A}\in\mathrm{D}$,then $\hat{A}$ is periodic with period T_0 ,if and only if Φ has a D- pullback absorbing set $\hat{B}\in\mathrm{D}$,and its period is T_0.

3. Existence and Uniqueness of Global Solution

Let $X=V_2\times H\times \mathrm{M}$, $z_0=(u_0,u_1,\eta_0)$, $z=z(t)=\big(u(t),u_t(t),\eta^t(s)\big)$.

Lemma2.1 Under assumption $(E_1)-(E_2)$,and $h\in L^2_{loc}(R,H)$, $z_0\in X$,Then z determined by problems(4)-(6)satisfies the following properties:

$$\|v\|^2+\|\Delta u\|^2+\|\eta^t\|^2_{\mathrm{M}}\le 4\mathrm{e}^{-\kappa_1\tau}H_1(t-\tau)+\frac{64\lambda_1^{-1}C_0^2|\Omega|}{\kappa_1}+16\lambda_1^{-1}\mathrm{e}^{-\kappa_1 t}\int_{t-\tau}^{t}\mathrm{e}^{\kappa_1 s}\|h(s)\|^2\,\mathrm{d}s,$$

$$\int_{t-\tau}^{t}\|\nabla v(s)\|^2\,\mathrm{d}s\le 2H_1(t-\tau)+32\lambda_1^{-1}C_0^2|\Omega|\tau+8\lambda_1^{-1}\int_{t-\tau}^{t}\|h(s)\|^2\,\mathrm{d}s .$$

where $v=u_t+\varepsilon u$, ε is a sufficiently small positive number, which will be given below.

Proof: Taking $H-$ inner product by $v=u_t+\varepsilon u$ in(4),we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2+\varepsilon^2\|u\|^2+\|\Delta u\|^2-\varepsilon\|\nabla u\|^2\right)+\|\nabla v\|^2-\varepsilon\|v\|^2+\varepsilon\|\Delta u\|^2-\varepsilon^2\|\nabla u\|^2+\varepsilon^3\|u\|^2+$$
$$\left(\int_0^\infty\mu(s)\Delta\left(a(x)\Delta\eta^t(s)\right)\mathrm{d}s,u_t\right)+\varepsilon\left(\int_0^\infty\mu(s)\Delta\left(a(x)\Delta\eta^t(s)\right)\mathrm{d}s,u\right)+\left(f(\sin u),v\right)=\left(h(x,t),v\right),$$
(8)

According to $\eta_t^t=-\eta_s^t+u_t$, Assumption (E_2) and Holder Inequality, we have

$$\left(\int_0^\infty\mu(s)\Delta\left(a(x)\Delta\eta^t(s)\right)\mathrm{d}s,u_t\right)\ge\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\eta^t\right\|_{\mathrm{M}}^2+\frac{\mu_1}{2}\left\|\eta^t\right\|_{\mathrm{M}}^2, \quad (9)$$

$$\varepsilon\left(\int_0^\infty\mu(s)\Delta\left(a(x)\Delta\eta^t(s)\right)\mathrm{d}s,u\right)\ge-\frac{\mu_1}{4}\left\|\eta^t\right\|_{\mathrm{M}}^2-\varepsilon^2\frac{\mu_0\|a\|_\infty}{\mu_1}\|\Delta u\|^2, \quad (10)$$

According to Assumption (E_1) and $h\in L_{loc}^2(R,H)$, we get

$$(h,v)\le\|h\|\|v\|\le2\lambda_1^{-1}\|h\|^2+\frac{1}{8}\|\nabla v\|^2, \quad (11)$$

$$\left(f(\sin u),v\right)\le\left\|f(\sin u)\right\|\|v\|\le2\lambda_1^{-1}\left\|f(\sin u)\right\|^2+\frac{1}{8}\|\nabla v\|^2\le8\lambda_1^{-1}C_0^2|\Omega|+\frac{1}{8}\|\nabla v\|^2, \quad (12)$$

Substituting(9)-(12)into(8),we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}H_1(t)+K_1(t)\le4\lambda_1^{-1}\|h\|^2+16\lambda_1^{-1}C_0^2|\Omega|, \quad (13)$$

where

$$H_1(t)=\|v\|^2+\varepsilon^2\|u\|^2+\|\Delta u\|^2-\varepsilon\|\nabla u\|^2+\left\|\eta^t\right\|_{\mathrm{M}}^2\ge\|v\|^2+\frac{1}{4}\|\Delta u\|^2+\left\|\eta^t\right\|_{\mathrm{M}}^2,$$

$$K_1(t)=\frac{3}{2}\|\nabla v\|^2-2\varepsilon\|v\|^2+2\varepsilon\|\Delta u\|^2+2\varepsilon^3\|u\|^2-2\varepsilon^2\|\nabla u\|^2$$
$$-\frac{2\mu_0\|a\|_\infty}{\mu_1}\varepsilon^2\|\Delta u\|^2+\frac{\mu_1}{2}\left\|\eta^t\right\|_{\mathrm{M}}^2\ge2\lambda_1^{-1}\|h\|^2+4\lambda_1^{-1}C|\Omega|\ge\left(\frac{3}{2}\lambda_1-2\varepsilon\right)\|v\|^2+2\varepsilon^3\|u\|^2-\varepsilon^2\|\nabla u\|^2$$
$$+\varepsilon\left(2-\varepsilon\lambda_1^{-1}-\frac{2\mu_0\|a\|_\infty}{\mu_1}\varepsilon\right)\|\Delta u\|^2+\frac{\mu_1}{2}\left\|\eta^t\right\|_{\mathrm{M}}^2+\frac{1}{2}\|\nabla v\|^2,$$

And take a sufficiently small ε, such that

$\frac{3}{2}\lambda_1-2\varepsilon>\varepsilon,\ 1-\varepsilon\lambda_1^{-1}-\frac{2\mu_0\|a\|_\infty}{\mu_1}\varepsilon>0,\ \varepsilon\left(1+\frac{\mu_0\|a\|_\infty}{2r_0}\right)\le\mu_1,\ \varepsilon+\frac{2\varepsilon+\varepsilon^2}{2r_0}\le2\lambda_1$,(r_0 will be given later, the last two conditions are given in advance).

Let $\kappa_1=\min\left\{\varepsilon,\frac{\mu_1}{2}\right\}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}H_1(t)+\kappa_1H_1(t)+\frac{1}{2}\|\nabla v\|^2\le4\lambda_1^{-1}\|h\|^2+16\lambda_1^{-1}C_0^2|\Omega|, \quad (14)$$

integrating(14) over $(t-\tau,t)$, we get

$$H_1(t)\le\mathrm{e}^{-\kappa_1\tau}H_1(t-\tau)+\frac{16\lambda_1^{-1}C_0^2|\Omega|}{\kappa_1}+4\lambda_1^{-1}\mathrm{e}^{-\kappa_1t}\int_{t-\tau}^t\mathrm{e}^{\kappa_1s}\|h(s)\|^2\,\mathrm{d}s, \quad (15)$$

$$\int_{t-\tau}^t\|\nabla v(s)\|^2\,\mathrm{d}s\le2H_1(t-\tau)+32\lambda_1^{-1}C_0^2|\Omega|\tau+8\lambda_1^{-1}\int_{t-\tau}^t\|h(s)\|^2\,\mathrm{d}s. \quad (16)$$

Then

$$\|v\|^2+\|\Delta u\|^2+\left\|\eta^t\right\|_{\mathrm{M}}^2\le4\mathrm{e}^{-\kappa_1\tau}H_1(t-\tau)+\frac{64\lambda_1^{-1}C_0^2|\Omega|}{\kappa_1}+16\lambda_1^{-1}\mathrm{e}^{-\kappa_1t}\int_{t-\tau}^t\mathrm{e}^{\kappa_1s}\|h(s)\|^2\,\mathrm{d}s, \quad (17)$$

$\int_{t-\tau}^{t}\|\nabla v(s)\|^2 \mathrm{d}s \le 2H_1(t-\tau)+32\lambda_1^{-1}C_0^2|\Omega|\tau+8\lambda_1^{-1}\int_{t-\tau}^{t}\|h(s)\|^2 \mathrm{d}s$. (18)

Lemma 2.1 is proved.

Theorem 2.1 Assume that $(E_1)-(E_2)$,and $h\in L_{loc}^2(R,X)$, $z_0\in X, T>0$,Then problem(4)-(6) admits a unique solution $z\in L^\infty([\tau,\infty),X), u_t\in L^2([\tau,T],H_0^1)$, and $z=(u(t),u_t(t),\eta^t(s))$ depends continuously on initial data in X .

Proof: We will use the faedo-Galerkin method to prove the existence of global solutions.

Step1.Construct approximate solution

Let $-\Delta\omega_i=\lambda_i\omega_i$,where λ_i is the eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary on Ω, ω_i is the characteristic function corresponding to eigenvalue λ_i ,According to the eigenvalue theory, $\omega_1,\omega_2,\cdots,\omega_l$ constitutes the standard orthogonal basis of.

Fixed $T>0$,for a given integer $m\in \mathrm{N}$,the projection operators from the following spaces to their subspaces are represented by P_m and Q_m ,respectively:

$span\{\omega_1,\omega_2,\cdots,\omega_l\}\subset V_2,\ span\{\varsigma_1,\varsigma_2,\cdots,\varsigma_l\}\subset \mathrm{M}$,

Let $u_{im}(t), \eta_{im}^t(s)$ be determined by the following nonlinear ordinary differential equations:

$$\begin{cases}\left(u_{mtt}-\Delta u_{mt}+\Delta^2 u_m+\int_0^\infty \mu(s)\Delta\left(a(x)\Delta\eta_m^t(s)\right)\mathrm{d}s+f(\sin u_m),\omega_i\right)=(h(x,t),\omega_i),\\ \left(\eta_{mt}^t+\eta_{ms}^t,\varsigma_i\right)_{\mathrm{M}}=(u_t,\varsigma_i)_{\mathrm{M}}, \qquad i=1,2,\cdots,m,\end{cases}\tag{19}$$

Where $u_m(t)=\sum_{i=1}^m u_{im}(t)\omega_i, \eta_m^t(s)=\sum_{i=1}^m \eta_{im}^t(s)\varsigma_i$,initial condition $z_{m0}=(u_{m0},u_{m1},\eta_{m0})$,when $m\to+\infty$, $z_{m0}\to z_0$ in X .From the basic theory of ordinary differential equation, we can know, the problem(4)-(6)has an approximate solution on $(\tau,T): z_m=(u_m,u_{mt},\eta_m^t)$.

Step2.priori estimation

To proof of existence of solution in X ,multiplying(19)by $u_{imt}(t)+\varepsilon u_{im}(t)$,And sum i ,let $v_m(t)=u_{mt}(t)+\varepsilon u_m(t)$,According to lemma 2.1, a priori estimate of solution in X space is obtained:

$$\|v_m\|^2+\|\Delta u_m\|^2+\|\eta_m^t\|_{\mathrm{M}}^2\le 4\mathrm{e}^{-\kappa_1\tau}H_{1m}(t-\tau)+\frac{64\lambda_1^{-1}C_0^2|\Omega|}{\kappa_1}+16\lambda_1^{-1}\mathrm{e}^{-\kappa_1 t}\int_{t-\tau}^{t}\mathrm{e}^{\kappa_1 s}\|h(s)\|^2\mathrm{d}s,\tag{20}$$

$$\int_{t-\tau}^{t}\|\nabla v_m(s)\|^2 \mathrm{d}s \le 2H_{1m}(t-\tau)+32\lambda_1^{-1}C_0^2|\Omega|\tau+8\lambda_1^{-1}\int_{t-\tau}^{t}\|h(s)\|^2 \mathrm{d}s\ .\tag{21}$$

So $z=(u_m,v_m,\eta_m^t)$ is uniformly bounded in $L^\infty([\tau,+\infty);X)$.

Step3.Limit process

From (20)(21),we get

Because $\{u_m\}$ is bounded on V_2 , $\{u_m\}$ has subsequences strongly convergent to u on,still denoted by $\{u_m\}$,such that

$\{u_m\}$ converges to u almost everywhere on H .

$v_m\to v$ in $L^\infty([\tau,+\infty);H)\cap L^2([\tau,T];H_0^1)$ weak*,

$u_m\to u$ in $L^\infty([\tau,+\infty);V_2)$ weak*,

$\eta_m^t\to\eta^t$ 在 $L^\infty([\tau,+\infty);\mathrm{M})$ weak*.

Then

Because of $(u_{mt},\omega_i)=(v_m,\omega_i)-(\varepsilon u_m,\omega_i)$, so $\left(u_{mt},\omega_i\right)\to\left(v,\omega_i\right)-\left(\varepsilon u,\omega_i\right)$ in $L^\infty[0,+\infty)$ weak*.

and $\left(-\Delta u_{mt},\omega_i\right)=\left(\nabla u_{mt},\nabla\omega_i\right)=\left(\nabla v_m,\lambda_i^{\frac{1}{2}}\omega_i\right)-\left(\varepsilon\nabla u_m,\lambda_i^{\frac{1}{2}}\omega_i\right)$,

then $\left(-\Delta u_{mt},\omega_i\right)\to\left(\nabla v,\lambda_i^{\frac{1}{2}}\omega_i\right)-\left(\varepsilon\nabla u,\lambda_i^{\frac{1}{2}}\omega_i\right)$ in $L^2[0,T]$ weak*.

According to $\left(u_{mt},\omega_i\right)\to\left(u_t,\omega_i\right)$ in $L^\infty[0,+\infty)$ weak*, $\left(u_{mtt},\omega_i\right)=\dfrac{\mathrm{d}}{\mathrm{d}t}\left(u_{mt},\omega_i\right)\to\left(u_{tt},\omega_i\right)$ in $D'[0,+\infty)$ weak,where $D'[0,+\infty)$ is the conjugate space of infinitely differentiable space $D[0,+\infty)$.

$\left(\Delta^2 u_m,\omega_i\right)=\left(-\Delta u_m,\lambda_i\omega_i\right)\to\left(-\Delta u,\lambda_i\omega_i\right)$ in $L^\infty[0,+\infty)$ weak*.

$\left(\int_0^\infty\mu(s)\Delta\left(a(x)\Delta\eta_m^t(s)\right)\mathrm{d}s,\omega_i\right)\to\left(\int_0^\infty\mu(s)\left(a(x)(-\Delta)\eta^t(s)\right)\mathrm{d}s,\lambda_i\omega_i\right)$ in $L^\infty[0,+\infty)$ weak*.

According to the assumption (E_1), $f(\sin u)$ is a continuous function,

We get $\left(f(\sin u_m),\omega_i\right)\to\left(f(\sin u),\omega_i\right)$ in $L^\infty[0,+\infty)$ weak*.

Based on the above convergence, $z_{m0}\to z_0$ in X weak. For all i and when $m\to+\infty$,according to the density of substrate $\omega_1,\omega_2,\cdots,\omega_l,\cdots$,we have

$$\left(u_{tt}-\Delta u_t+\Delta^2 u+\int_0^\infty\mu(s)\Delta\left(a(x)\Delta\eta^t(s)\right)\mathrm{d}s+f(\sin u),\omega\right)=\left(h(x,t),\omega\right),\ \forall\omega\in V_2,$$

$$\left(\eta_t^t+\eta_s^t,\varsigma\right)_{\mathrm{M}}=\left(u_t,\varsigma\right)_{\mathrm{M}},\ \forall\varsigma\in\mathrm{M}。$$

Therefore,the existence of weak global solutions to problem(4)-(6)is proved.

The uniqueness and continuous dependence of the solution:

Let $z_i=\left(u_i(t),u_{it}(t),\eta_i^t(s)\right),(i=1,2)$, $\forall t\in R$, be two solutions of problem(4)-(6)as shown above corresponding to initial data $z_0^i=\left(u_0^i,u_1^i,\eta_0^i(s)\right)$,Then $w(t)=u_1(t)-u_2(t)$, $\xi^t(s)=\eta_1^t(s)-\eta_2^t(s)$ satisfifies

$$\begin{cases}w_{tt}-\Delta w_t+\Delta^2 w+\int_0^\infty\mu(s)\Delta\left(a(x)\Delta\xi^t(x,s)\right)\mathrm{d}s+f(\sin u_1)-f(\sin u_2)=0,\\ \xi_t=-\xi_s+w_t,\\ \left(w(\tau),w_t(\tau),\xi^\tau(s)\right)=\left(u_0^1,u_1^1,\eta_0^1\right)-\left(u_0^2,u_1^2,\eta_0^2\right),\end{cases}\tag{22}$$

Taking H-inner product by w_t in(22) and making use of assumptions $(E_1)(E_2)$,we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|w_t\|^2+\|\Delta w\|^2+\frac{1}{2}\|\xi^t\|_{\mathrm{M}}^2\right]+\|\nabla w_t\|^2+\frac{\mu_1}{2}\|\xi^t\|_{\mathrm{M}}^2\le-\left(f(\sin u_1)-f(\sin u_2),w_t\right)$$

$$\le\left|\left(f'(\alpha)(\sin u_1-\sin u_2),w_t\right)\right|\le2\left|\left(f'(\alpha)\cos\frac{u_1+u_2}{2}\sin\frac{u_1-u_2}{2},w_t\right)\right|\tag{23}$$

$$\le2qC_0\|w\|\|w_t\|\le qC_0\|w\|^2+qC_0\|w_t\|,$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|w_t\|^2+\|\Delta w\|^2+\|\xi^t\|_{\mathrm{M}}^2\right]+2\|\nabla w_t\|^2\le\beta\left(\|w_t\|^2+\|\Delta w\|^2+\|\xi^t\|_{\mathrm{M}}^2\right),\tag{24}$$

where $\beta=\max\left\{2qC_0\lambda_1^{-2},2qC_0,\mu_1\right\}$.

Applying the Gronwall inequality to(24) yields

$$\|w_t(t)\|^2+\|\Delta w(t)\|^2+\|\xi^t\|_{\mathrm{M}}^2\le\left(\|w_t(\tau)\|^2+\|\Delta w(\tau)\|^2+\|\xi^\tau\|_{\mathrm{M}}^2\right)\mathrm{e}^{\beta(t-\tau)}.\tag{25}$$

(25) implies that $z=\left(u(t),u_t(t),\eta^t(s)\right)$ depends continuously on initial data in X ,and hence, the

solution of problem(4)-(6) is unique. Then the uniqueness and continuous dependence of the solution are proved.

Theorem 2.1 is proven.

4. Existence of Pullback Attractor

Let $\Omega_1 = R,\ \theta_t(\tau) = \tau + t,\ \forall \tau \in R$,and

$\Phi(t,\tau,z_0) = z(t+\tau,\tau,z_0) = \left(u(t+\tau), u_t(t+\tau), \eta^{t+\tau}(s)\right), \tau \in R, t \ge \tau, z_0 \in X$. (26)

From the existence and uniqueness of the solution of problem(4)-(6),we have

$\Phi(t+r,\tau,z_0) = \Phi\left(t,\tau+r,\Phi(r,\tau,z_0)\right), \tau \in R, t,r \ge \tau, z_0 \in X$,

And for any $\tau \in R, t \ge \tau$, $\Phi(t,\tau,\bullet): X \to X$ is continuous,so, Φ is a continuous cocycle.

Assume that $h(x,t) \in L^2_{loc}(R;H)$ and

$\int_{-\infty}^{t} \mathrm{e}^{\sigma s} \left\| h(\bullet,s) \right\|^2 \mathrm{d}s < \infty, \forall t \in R$, (27)

where $\sigma = \kappa_1$.

Let $r^2(t) = \int_{-\infty}^{0} \mathrm{e}^{\sigma s} \left\| h(\bullet, s+t) \right\|^2 \mathrm{d}s$, R_σ be the set of all functions $r: R \to (0,+\infty)$ such that

$\lim\limits_{t\to-\infty} \mathrm{e}^{\sigma t} r^2(t) = 0$, (28)

And denote by D_σ the class of all families 为 $\hat{D} = \left\{D(t); t \in R\right\} \subset \mathrm{P}(X)$ such that $D(t) \subset \overline{B}\left(0, r_{\hat{D}}(t)\right)$,

For some $r_{\hat{D}} \in \mathrm{R}_\sigma$,where $\overline{B}\left(0, r_{\hat{D}}(t)\right)$ denotes the closed ball in X centered at 0 with radius $r_{\hat{D}}(t)$.

By lemma2.1,and(27),we get

$\left\| \Phi(\tau,\ t-\tau, z_{t-\tau}) \right\|_X^2 = \left\| z(t,\ t-\tau, z_{t-\tau}) \right\|_X^2 = \left\| \left(u(t),\ u_t(t), \eta^t(s)\right) \right\|_X^2$ is bounded.

Let $R_\sigma(t) = \dfrac{64\lambda_1^{-1} C_0^2 |\Omega|}{\sigma} + 16\lambda_1^{-1} \mathrm{e}^{-\sigma t} \int_{t-\tau}^{t} \mathrm{e}^{\sigma s} \left\| h(s) \right\|^2 \mathrm{d}s,$

and $B_\sigma(t) = \left\{ z \in X; \|z\|_X^2 \le R_\sigma(t) \right\}$,it is straightforward to check that $\hat{B}_\sigma \in \mathrm{D}_\sigma$,and moreover the family $\hat{B}_\sigma$ is D_σ - pullback absorbing for the cocycle Φ .

Lemma3.1 Assumed that the assumptions of Theorem2.1,and $h \in L^2_{loc}(R, X)$ satisfies (27),the cocycle Φ has

a D_σ - pullback absorbing set in X .

Lemma3.2 Assumed that the assumptions of Theorem2.1,and $h \in L^2_{loc}(R, X)$ satisfies (27),the cocycle Φ is

D_σ - pullback asymptotically compact.

Proof:For $t_0 \in R$, let $z_i = \left(u_i(t), u_{it}(t), \eta_i^t(s)\right), (i=1,2)$ be two solutions of problem(4)-(6) as shown above corresponding to initial data $z_0^i = \left(u_0^i, u_1^i, \eta_0^i(s)\right) \in D(t_0 - \tau)$,where $\tau \ge 0$,

Then $w(t) = u_1(t) - u_2(t),\ \xi^t(s) = \eta_1^t(s) - \eta_2^t(s)$ satisfifies

$$\begin{cases} w_{tt} - \Delta w_t + \Delta^2 w + \int_0^\infty \mu(s) \Delta\left(a(x) \Delta \xi^t(s)\right) \mathrm{d}s + f(\sin u_1) - f(\sin u_2) = 0, \\ \xi_t = -\xi_s + w_t,\ w|_\Gamma = \Delta w|_\Gamma = 0, \\ \left(w_0, w_1, \xi_0(s)\right) = \left(u_0^1, u_1^1, \eta_0^1\right) - \left(u_0^2, u_1^2, \eta_0^2\right), \end{cases} \tag{29}$$

Taking H-inner product by $\mathrm{e}^{rt}w_t$ in (29) (where $r=\sigma$),and making use of assumptions $(E_1)-(E_2)$,we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{rt}H_w(t)\right)+2\mathrm{e}^{rt}\|\nabla w_t\|^2+\mu_1\mathrm{e}^{rt}\|\xi^t\|_{\mathrm{M}}^2\le r\mathrm{e}^{rt}H_w(t)-2\left(f(\sin u_1)-f(\sin u_2),\mathrm{e}^{rt}w_t\right),\quad (30)$$

here $H_w(t)=\|w_t\|^2+\|\Delta w\|^2+\|\xi^t\|_{\mathrm{M}}^2$.

integrating (30)over the interval$[s,t_0]$,we obtain

$$\begin{aligned}&\mathrm{e}^{rt_0}H_w(t_0)-\mathrm{e}^{rs}H_w(s)+2\int_s^{t_0}\mathrm{e}^{r\zeta}\|\nabla w_t(\zeta)\|^2\mathrm{d}\zeta+\mu_1\int_s^{t_0}\mathrm{e}^{r\zeta}\|\xi^\zeta\|_{\mathrm{M}}^2\mathrm{d}\zeta\\&\le r\int_s^{t_0}\mathrm{e}^{r\zeta}H_w(\zeta)\mathrm{d}\zeta-2\int_s^{t_0}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),\mathrm{e}^{r\zeta}w_t(\zeta)\right)\mathrm{d}\zeta\quad (31)\\&\le r\int_s^{t_0}\mathrm{e}^{r\zeta}H_w(\zeta)\mathrm{d}\zeta+2\int_s^{t_0}\left(f(\sin u_2(\zeta))-f(\sin u_1(\zeta)),\mathrm{e}^{r\zeta}w_t(\zeta)\right)\mathrm{d}\zeta,\end{aligned}$$

integrating(31) over the interval $[t_0-\tau,t_0]$ with respect to s,we have

$$\begin{aligned}&\tau\mathrm{e}^{rt_0}H_w(t_0)-\int_{t_0-\tau}^{t_0}\mathrm{e}^{rs}H_w(s)\mathrm{d}s+2\int_{t_0-\tau}^{t_0}\int_s^{t_0}\mathrm{e}^{r\zeta}\|\nabla w_t(\zeta)\|^2\mathrm{d}\zeta\mathrm{d}s+\mu_1\int_{t_0-\tau}^{t_0}\int_s^{t_0}\mathrm{e}^{r\zeta}\|\xi^\zeta\|_{\mathrm{M}}^2\mathrm{d}\zeta\mathrm{d}s\\&\le r\int_{t_0-\tau}^{t_0}\int_s^{t_0}\mathrm{e}^{r\zeta}H_w(\zeta)\mathrm{d}\zeta\mathrm{d}s+2\int_{t_0-\tau}^{t_0}\int_s^{t_0}\mathrm{e}^{r\zeta}\left(f(\sin u_2(\zeta))-f(\sin u_1(\zeta)),w_t(\zeta)\right)\mathrm{d}\zeta\mathrm{d}s,\end{aligned}\quad (32)$$

Taking H-inner product by $\mathrm{e}^{rt}w$ in (29),we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{rt}(w_t,w)+\frac{1}{2}\mathrm{e}^{rt}\|\nabla w\|^2\right)+\mathrm{e}^{rt}\|\Delta w\|^2\le r\mathrm{e}^{rt}(w_t,w)+\frac{1}{2}r\mathrm{e}^{rt}\|\nabla w\|^2+\mathrm{e}^{rt}\|w_t\|^2\\&+\frac{\mu_0\|a\|_\infty}{2}\mathrm{e}^{rt}\|\xi^t\|_{\mathrm{M}}^2+\frac{1}{2}\mathrm{e}^{rt}\|\Delta w\|^2-\mathrm{e}^{rt}\left(f(\sin u_1)-f(\sin u_2),w\right),\end{aligned}\quad (33)$$

integrating(33) over the interval$[s,t_0]$,we arrive at

$$\begin{aligned}&\left(\frac{1}{2}-\frac{1}{2}\lambda_1^{-1}r-\frac{1}{2}\lambda_1^{-2}r\right)\int_s^{t_0}\mathrm{e}^{r\zeta}\|\Delta w(\zeta)\|^2\mathrm{d}\zeta+\mathrm{e}^{rt_0}\left(w_t(t_0),w(t_0)\right)+\frac{1}{2}\mathrm{e}^{rt_0}\|\nabla w(t_0)\|^2\\&\le\mathrm{e}^{rs}\left(w_t(s),w(s)\right)+\frac{1}{2}\mathrm{e}^{rs}\|\nabla w(s)\|^2+\left(1+\frac{r}{2}\right)\int_s^{t_0}\mathrm{e}^{r\zeta}\|w_t(\zeta)\|^2\mathrm{d}\zeta\quad (34)\\&+\frac{\mu_0\|a\|_\infty}{2}\int_s^{t_0}\mathrm{e}^{r\zeta}\|\xi^\zeta\|_{\mathrm{M}}^2\mathrm{d}\zeta-\int_s^{t_0}\mathrm{e}^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w(\zeta)\right)\mathrm{d}\zeta,\end{aligned}$$

integrating(34) over the interval $[t_0-\tau,t_0]$ with respect to s,we have

$$\begin{aligned}&\left(\frac{1}{2}-\frac{1}{2}\lambda_1^{-1}r-\frac{1}{2}\lambda_1^{-2}r\right)\int_{t_0-\tau}^{t_0}\int_s^{t_0}\mathrm{e}^{r\zeta}\|\Delta w(\zeta)\|^2\mathrm{d}\zeta\mathrm{d}s\le\int_{t_0-\tau}^{t_0}\mathrm{e}^{rs}\left(w_t(s),w(s)\right)\mathrm{d}s+\frac{1}{2}\int_{t_0-\tau}^{t_0}\mathrm{e}^{rs}\|\nabla w(s)\|^2\mathrm{d}s\\&+\left(1+\frac{r}{2}\right)\int_{t_0-\tau}^{t_0}\int_s^{t_0}\mathrm{e}^{r\zeta}\|w_t(\zeta)\|^2\mathrm{d}\zeta\mathrm{d}s-\tau\mathrm{e}^{rt_0}\left(w_t(t_0),w(t_0)\right)-\frac{\tau}{2}\mathrm{e}^{rt_0}\|\nabla w(t_0)\|^2\\&+\frac{\mu_0\|a\|_\infty}{2}\int_{t_0-\tau}^{t_0}\int_s^{t_0}\mathrm{e}^{r\zeta}\|\xi^\zeta\|_{\mathrm{M}}^2\mathrm{d}\zeta\mathrm{d}s-\int_{t_0-\tau}^{t_0}\int_s^{t_0}\mathrm{e}^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w(\zeta)\right)\mathrm{d}\zeta\mathrm{d}s,\end{aligned}$$

(35)

Let $r_0=\frac{1}{2}-\frac{1}{2}\lambda_1^{-1}r-\frac{1}{2}\lambda_1^{-2}r>0$.

Substituting(35)into(32),then

$$\tau e^{rt_0}H_w(t_0)-\int_{t_0-\tau}^{t_0} e^{rs}H_w(s)\mathrm{d}s \le r\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{r\zeta}\|w_t(\zeta)\|^2 \mathrm{d}\zeta\mathrm{d}s + r\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{r\zeta}\|\xi^\zeta\|_{\mathrm{M}}^2 \mathrm{d}\zeta\mathrm{d}s$$
$$+\frac{r}{r_0}\int_{t_0-\tau}^{t_0} e^{rs}\left(w_t(s),w(s)\right)\mathrm{d}s+\frac{r}{2r_0}\int_{t_0-\tau}^{t_0} e^{rs}\|\nabla w(s)\|^2 \mathrm{d}s$$
$$+\frac{2r+r^2}{2r_0}\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{r\zeta}\|w_t(\zeta)\|^2 \mathrm{d}\zeta\mathrm{d}s-\tau\frac{r}{r_0}e^{rt_0}\left(w_t(t_0),w(t_0)\right)-\tau\frac{r}{2r_0}e^{rt_0}\|\nabla w(t_0)\|^2$$
$$+\mu_0\frac{r}{2r_0}\|a\|_\infty\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{r\zeta}\|\xi^\zeta\|_{\mathrm{M}}^2 \mathrm{d}\zeta\mathrm{d}s-\frac{r}{r_0}\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w(\zeta)\right)\mathrm{d}\zeta\mathrm{d}s$$
$$-2\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{r\zeta}\|\nabla w_t(\zeta)\|^2 \mathrm{d}\zeta\mathrm{d}s-\mu_1\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{r\zeta}\|\xi^\zeta\|_{\mathrm{M}}^2 \mathrm{d}\zeta\mathrm{d}s$$
$$-2\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w_t(\zeta)\right)\mathrm{d}\zeta\mathrm{d}s$$
$$\le\frac{r}{r_0}\int_{t_0-\tau}^{t_0} e^{rs}\left(w_t(s),w(s)\right)\mathrm{d}s+\frac{r}{2r_0}\int_{t_0-\tau}^{t_0} e^{rs}\|\nabla w(s)\|^2 \mathrm{d}s$$
$$-\frac{r}{r_0}\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w(\zeta)\right)\mathrm{d}\zeta\mathrm{d}s \quad (36)$$
$$-2\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w_t(\zeta)\right)\mathrm{d}\zeta\mathrm{d}s$$
$$-\tau\frac{r}{r_0}e^{rt_0}\left(w_t(t_0),w(t_0)\right)-\tau\frac{r}{2r_0}e^{rt_0}\|\nabla w(t_0)\|^2,$$

integrating(33) over the interval $[t_0-\tau,t_0]$,we get

$$\left(\frac{1}{2}-\frac{r}{2\lambda_1}\right)\int_{t_0-\tau}^{t_0} e^{r\zeta}\|\Delta w(\zeta)\|^2 \mathrm{d}\zeta + e^{rt_0}\left(w_t(t_0),w(t_0)\right)+\frac{1}{2}e^{rt_0}\|\nabla w(t_0)\|^2 \le$$
$$e^{r(t_0-\tau)}\left(w_t(t_0-\tau),w(t_0-\tau)\right)+\frac{1}{2}e^{r(t_0-\tau)}\|\nabla w(t_0-\tau)\|^2+r\int_{t_0-\tau}^{t_0} e^{r\zeta}\left(w_t(\zeta),w(\zeta)\right)\mathrm{d}\zeta+$$
$$\int_{t_0-\tau}^{t_0} e^{r\zeta}\|w_t(\zeta)\|^2 \mathrm{d}\zeta+\frac{\mu_0\|a\|_\infty}{2}\int_{t_0-\tau}^{t_0} e^{r\zeta}\|\xi^\zeta\|_{\mathrm{M}}^2 \mathrm{d}\zeta-\int_{t_0-\tau}^{t_0} e^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w(\zeta)\right)\mathrm{d}\zeta,$$

(37)

Substituting (37)into(36),we receive

$$\tau e^{rt_0}H_w(t_0)+\int_{t_0-\tau}^{t_0} e^{rs}H_w(s)\mathrm{d}s \le 6\int_{t_0-\tau}^{t_0} e^{r\zeta}\|w_t(\zeta)\|^2 \mathrm{d}\zeta+\left(2+2\mu_0\|a\|_\infty\right)\int_{t_0-\tau}^{t_0} e^{r\zeta}\|\xi^\zeta\|_{\mathrm{M}}^2 \mathrm{d}\zeta$$
$$+4e^{r(t_0-\tau)}\left(w_t(t_0-\tau),w(t_0-\tau)\right)-4e^{rt_0}\left(w_t(t_0),w(t_0)\right)$$
$$+2e^{r(t_0-\tau)}\|\nabla w(t_0-\tau)\|^2-2e^{rt_0}\|\nabla w(t_0)\|^2$$
$$+\left(\frac{r}{r_0}+4r\right)\int_{t_0-\tau}^{t_0} e^{r\zeta}\left(w_t(\zeta),w(\zeta)\right)\mathrm{d}\zeta-4\int_{t_0-\tau}^{t_0} e^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w(\zeta)\right)\mathrm{d}\zeta \quad (38)$$
$$+\frac{r}{2r_0}\int_{t_0-\tau}^{t_0} e^{r\zeta}\|\nabla w(\zeta)\|^2 \mathrm{d}\zeta-\frac{r}{r_0}\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w(\zeta)\right)\mathrm{d}\zeta\mathrm{d}s$$
$$-2\int_{t_0-\tau}^{t_0}\int_s^{t_0} e^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w_t(\zeta)\right)\mathrm{d}\zeta\mathrm{d}s$$
$$-\tau\frac{r}{r_0}e^{rt_0}\left(w_t(t_0),w(t_0)\right)-\tau\frac{r}{2r_0}e^{rt_0}\|\nabla w(t_0)\|^2,$$

also,integrating(30) over the interval $[t_0-\tau,t_0]$,we get

$$\mathrm{e}^{rt_0}H_w(t_0)-\mathrm{e}^{r(t_0-\tau)}H_w(t_0-\tau)+2\int_{t_0-\tau}^{t_0}\mathrm{e}^{r\zeta}\left\|\nabla w_t(\zeta)\right\|^2\mathrm{d}\zeta+\mu_1\int_{t_0-\tau}^{t_0}\mathrm{e}^{r\zeta}\left\|\xi^\zeta\right\|_{\mathrm{M}}^2\mathrm{d}\zeta$$
$$\le r\int_{t_0-\tau}^{t_0}\mathrm{e}^{r\zeta}H_w(\zeta)\mathrm{d}\zeta-2\int_{t_0-\tau}^{t_0}\mathrm{e}^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w_t(\zeta)\right)\mathrm{d}\zeta, \tag{39}$$

Substituting (39)into(38),we receive

$$\begin{aligned}
\tau\mathrm{e}^{rt_0}H_w(t_0)&\le\frac{b}{r}\mathrm{e}^{r(t_0-\tau)}H_w(t_0-\tau)-\frac{2b}{r}\int_{t_0-\tau}^{t_0}\mathrm{e}^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w_t(\zeta)\right)\mathrm{d}\zeta\\
&+4\mathrm{e}^{r(t_0-\tau)}\left(w_t(t_0-\tau),w(t_0-\tau)\right)-4\mathrm{e}^{rt_0}\left(w_t(t_0),w(t_0)\right)\\
&+2\mathrm{e}^{r(t_0-\tau)}\left\|\nabla w(t_0-\tau)\right\|^2-2\mathrm{e}^{rt_0}\left\|\nabla w(t_0)\right\|^2\\
&+\left(\frac{r}{r_0}+4r\right)\int_{t_0-\tau}^{t_0}\mathrm{e}^{r\zeta}\left(w_t(\zeta),w(\zeta)\right)\mathrm{d}\zeta-4\int_{t_0-\tau}^{t_0}\mathrm{e}^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w(\zeta)\right)\mathrm{d}\zeta\\
&+\frac{r}{r_0}\int_{t_0-\tau}^{t_0}\mathrm{e}^{r\zeta}\left\|\nabla w(\zeta)\right\|^2\mathrm{d}\zeta-\frac{r}{r_0}\int_{t_0-\tau}^{t_0}\int_s^{t_0}\mathrm{e}^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w(\zeta)\right)\mathrm{d}\zeta\mathrm{d}s\\
&-2\int_{t_0-\tau}^{t_0}\int_s^{t_0}\mathrm{e}^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w_t(\zeta)\right)\mathrm{d}\zeta\mathrm{d}s\\
&-\tau\frac{r}{r_0}\mathrm{e}^{rt_0}\left(w_t(t_0),w(t_0)\right)-\tau\frac{r}{2r_0}\mathrm{e}^{rt_0}\left\|\nabla w(t_0)\right\|^2,
\end{aligned} \tag{40}$$

where $b=\max\left\{3r\lambda_1^{-1},\frac{2+2\mu_0\|a\|_\infty}{\mu_1}r\right\}$.

Combined with the above estimation, and combined with(40),let

$$\begin{aligned}
\Psi_{t_0,\tau}\left((u_0^1,u_1^1,\eta_0^1),(u_0^2,u_1^2,\eta_0^2)\right)&=\frac{r}{2r_0}\tau^{-1}\mathrm{e}^{-rt_0}\int_{t_0-\tau}^{t_0}\mathrm{e}^{r\zeta}\left\|\nabla w(\zeta)\right\|^2\mathrm{d}\zeta\\
&+\left(\frac{r}{r_0}+4r\right)\tau^{-1}\mathrm{e}^{-rt_0}\int_{t_0-\tau}^{t_0}\mathrm{e}^{r\zeta}\left(w_t(\zeta),w(\zeta)\right)\mathrm{d}\zeta\\
&-\frac{2b}{r}\tau^{-1}\mathrm{e}^{-rt_0}\int_{t_0-\tau}^{t_0}\mathrm{e}^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w_t(\zeta)\right)\mathrm{d}\zeta\\
&-4\tau^{-1}\mathrm{e}^{-rt_0}\int_{t_0-\tau}^{t_0}\mathrm{e}^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w(\zeta)\right)\mathrm{d}\zeta\\
&-\frac{r}{r_0}\tau^{-1}\mathrm{e}^{-rt_0}\int_{t_0-\tau}^{t_0}\int_s^{t_0}\mathrm{e}^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w(\zeta)\right)\mathrm{d}\zeta\mathrm{d}s\\
&-2\tau^{-1}\mathrm{e}^{-rt_0}\int_{t_0-\tau}^{t_0}\int_s^{t_0}\mathrm{e}^{r\zeta}\left(f(\sin u_1(\zeta))-f(\sin u_2(\zeta)),w_t(\zeta)\right)\mathrm{d}\zeta\mathrm{d}s\\
&-\left(\frac{r}{r_0}+4\tau^{-1}\right)\left(w_t(t_0),w(t_0)\right)-\left(\frac{r}{2r_0}+2\lambda_1\tau^{-1}\right)\left\|w(t_0)\right\|^2,
\end{aligned} \tag{41}$$

and

$$\tilde{R}_{t_0,\tau}=\frac{b}{r}H_w(t_0-\tau)+4\left(w_t(t_0-\tau),w(t_0-\tau)\right)+2\left\|\nabla w(t_0-\tau)\right\|^2,$$

then

$$H_w(t_0)\le\tau^{-1}\mathrm{e}^{-r\tau}\tilde{R}_{t_0,\tau}+\Psi_{t_0,\tau}\left((u_0^1,u_1^1,\eta_0^1),(u_0^2,u_1^2,\eta_0^2)\right).$$

By $\lim_{\tau\to\infty}\tau^{-1}e^{-r\tau}\tilde{R}_{t_0,\tau}=0$,for $\forall\,\nu>0$,we take τ_0 large enough such that

$$H_w(t_0)\le\nu+\Psi_{t_0,\tau_0}\left((u_0^1,u_1^1,\eta_0^1),(u_0^2,u_1^2,\eta_0^2)\right).$$

Thanks to Theorem1.2,Theorem2.1 and lemma3.1,it is sufficiently to prove that the function

$\Psi_{t_0,\tau_0}(\bullet,\bullet)\in contr(B_\sigma)$,let$(u_n(t),u_{nt}(t),\eta_n)$ be the corresponding solution of$(u_0^n,u_1^n,\eta_0^n)\in B_\sigma$ $(n=1,2,\cdots)$,from the observation above, without loss of generality(or by passing to subsequences),we assume that

① $u_n\to u$ $*$weakly in $L^\infty(t_0-\tau_0,t_0;V_2)$,

② $u_{nt}\to u_t$ $*$weakly in $L^\infty(t_0-\tau_0,t_0;H)$,

③ $u_n\to u$ in $L^2(t_0-\tau_0,t_0;H)$,

④ $u_n(t_0-\tau_0)\to u(t_0-\tau_0)$, $u_n(t_0)\to u(t_0)$ in $L^2(\Omega)$.

Now,we will deal with each term in (41)one by one, is a bounded subset of X ,Assumed that the assumptions of ①②④,易得

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{t_0-\tau_0}^{t_0}\mathrm{e}^{r\zeta}\left\|u_m(\zeta)-u_n(\zeta)\right\|^2\mathrm{d}\zeta=0\text{, (42)}$$

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{t_0-\tau_0}^{t_0}\mathrm{e}^{r\zeta}\left\|\nabla u_m(\zeta)-\nabla u_n(\zeta)\right\|^2\mathrm{d}\zeta=0\text{, (43)}$$

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{t_0-\tau_0}^{t_0}\int_\Omega\left(u_{mt}(\zeta)-u_{nt}(\zeta)\right)\bullet\left(u_m(\zeta)-u_n(\zeta)\right)\mathrm{d}x\mathrm{d}\zeta=0\text{, (44)}$$

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_\Omega\left(u_{mt}(t_0)-u_{nt}(t_0)\right)\bullet\left(u_m(t_0)-u_n(t_0)\right)\mathrm{d}x=0\text{, (45)}$$

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_\Omega\left|u_m(t_0)-u_n(t_0)\right|^2\mathrm{d}x=0\text{, (46)}$$

from(I),we further have

$$\begin{aligned}&\left|f(\sin u_m(s))-f(\sin u_n(s))\right|=\left|f'(\varpi)(\sin u_m(s)-\sin u_n(s))\right|\\&=\left|2f'(\varpi)\cos\frac{u_m(s)+u_n(s)}{2}\sin\frac{u_m(s)-u_n(s)}{2}\right|\le C_0q\left|u_m(s)-u_n(s)\right|,\end{aligned}\quad(47)$$

from③ ,we obtain that

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{t_0-\tau_0}^{t_0}e^{rs}\int_\Omega\left(f(\sin u_m(s))-f(\sin u_n(s))\right)\left(u_m(s)-u_n(s)\right)\mathrm{d}x\mathrm{d}s=0\text{, (48)}$$

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{t_0-\tau_0}^{t_0}\int_s^{t_0}\mathrm{e}^{r\zeta}\int_\Omega\left(f(\sin u_m(\zeta))-f(\sin u_n(\zeta))\right)\left(u_m(\zeta)-u_n(\zeta)\right)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s=0\text{, (49)}$$

by(44)(45),we get

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{t_0-\tau_0}^{t_0}\mathrm{e}^{rs}\int_\Omega\left(f(\sin u_m(s))-f(\sin u_n(s))\right)\left(u_{mt}(s)-u_{nt}(s)\right)\mathrm{d}x\mathrm{d}s=0\text{, (50)}$$

For each fixed t_0, $\left|\int_s^{t_0}\mathrm{e}^{r\zeta}\int_\Omega\left(f(\sin u_m(\zeta))-f(\sin u_n(\zeta))\right)\left(u_{mt}(\zeta)-u_{nt}(\zeta)\right)\mathrm{d}x\mathrm{d}\zeta\right|$ is bounded,by the Lebesgue dominated convergence theorem,we finally get

$$\begin{aligned}&\lim_{m\to\infty}\lim_{n\to\infty}\int_{t-\tau_0}^{t}\int_s^t\mathrm{e}^{r\zeta}\int_\Omega\left(f(\sin u_m(\zeta))-f(\sin u_n(\zeta))\right)\left(u_{mt}(\zeta)-u_{nt}(\zeta)\right)\mathrm{d}x\mathrm{d}\zeta\mathrm{d}s\\&=\int_{t-\tau_0}^{t}\left(\lim_{m\to\infty}\lim_{n\to\infty}\int_s^t\mathrm{e}^{r\zeta}\int_\Omega\left(f(\sin u_m(\zeta))-f(\sin u_n(\zeta))\right)\left(u_{mt}(\zeta)-u_{nt}(\zeta)\right)\mathrm{d}x\mathrm{d}\zeta\right)\mathrm{d}s=0,\end{aligned}\quad(51)$$

Hence,from(42)-(51),we see that $\Psi_{t_0,\tau_0}(\bullet,\bullet)\in contr(B_\sigma)$.

Lemma 3.2 is proven.

The main conclusions of this paper are as follows

Theorem3.1 In addition to the assumptions of Theorem2.1,and $h\in L^2_{loc}(R,X)$ satisfies (27),the cocycle Φ has a D_σ- pullback attractor in X .

Proof:From theorem 1.1, theorem 1.2, lemma 3.1 and lemma 3.2, we can get the conclusion of theorem 3.1.

Theorem3.2 In addition to the assumptions of Theorem2.1,and $h\in L^2_{loc}(R,X)$ satisfies (27),also

$h(x,t)$ is a periodic function with period T_0 ,then the cocycle Φ has a D_σ - pullback attractor in X which is periodic

with period T_0 .

Proof:In addition to the assumptions of Theorem2.1,and $h \in L^2_{loc}(R, X)$ satisfies (27),also $h(x,t)$ is a periodic function with period T_0 ,then for every $\bar{z} \in X, t \geq 0, \tau \in R$,we have that

$\Phi(t, \tau+T_0, \bar{z}) = z(t+\tau+T_0, \tau+T_0, \bar{z}) = z(t+\tau, \tau, \bar{z}) = \Phi(t, \tau, \bar{z})$,

By Defifinition1.2,we fifind that Φ is periodic with period T_0 ,Let $\hat{D} \in \mathrm{D}_\sigma$ and $\hat{D}_T$ be the $T_0 -$ translation of $\hat{D}$,Then for every $\sigma > 0, s \in R$,we have

$\lim_{r\to\infty} \mathrm{e}^{-\sigma r} \left\|\hat{D}(s-r)\right\|^2 = 0$, (45)

In particular,when $s = \tau + T_0, \tau \in R$,we get from(45)that

$\lim_{r\to\infty} \mathrm{e}^{-\sigma r} \left\|\hat{D}_T(s-r)\right\|^2 = \lim_{r\to\infty} \mathrm{e}^{-\sigma r} \left\|\hat{D}(\tau+T_0-r)\right\|^2 = 0$, (46)

From(46),we have $\hat{D}_T \in \mathrm{D}_\sigma$,and D_σ is $T_0 -$ translation closed. Similarly, one may check that D_σ is also $T_0 -$ translation closed.

Therefore, D_σ is $T_0 -$ translation invariant.

From theorem 3.1 and theorem 1.3, we can get the conclusion of theorem 3.2.

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References

[1] Cavalcanti M M, Fatori L H and Ma T F. Attractors for wave equations with degenerate memory[J]. Journal of Differential Equations, 2016. 260(1): 56-83.

[2] Lin G G, Lv P and Lou R J. Exponential attractors and inertial manifolds for a class of nonlinear generalized Kirchhoff-Boussinesq model[J]. Far East Journal of Mathematical Sciences, 2017. 101(9): 1913-1945.

[3] Peng Xiaoming, Zheng Xiaoxiao,Shang Yadong. Global Attractors for a Fourth Order Pseudo-Parabolic Equation with Fading Memory[J]. Acta Mathematica Scientia, 2019. 39(01): 114-124.

[4] ZHANG Hong,WANG Danxia. Global solution in Non-linear Boundary Conditions for Kirchhoff-Beam Equation with Memory Term[J].Mathematica Applicata, 2015. 028(2): 388-394.

[5] Zhang Jian-Wen,Ren Yong-Hua,Wu Run-Heng,Feng Tao. The global attractor of nonlinear thermoelastic coupled Sine-Gordon system[J]. Acta Physica Sinica, 2012. 61(11): 117-121.

[6] XU Qian,LI Xiao-jun. The existence of pullback attractors for non-autonomous wave equations with nonlinear damping[J]. Journal of Natural Science of HeiLongJiang University, 2014. 31(03): 306-314.

[7] WANG Fangping,Ma Qiaozhen, Existence of Pullback Attractors for Weakly Damped Suspension Bridge Equations with Linear Memory[J]. Journal of Jilin University(Science Edition), 2018. 56(06): 29-36.

[8] Qin Y, Yang X and Liu X. Pullback attractors for the non-autonomous Benjamin-Bona-Mahony equations in H2[J]. Acta Mathematica Scientia(English Series), 2012. 32(4): 1338–1348.

[9] NING D, Z. XIAO-PENG and G. Wen-jie. Pullback Attractor of a Non-autonomous Model for Epitaxial Growth[J]. Communications in Mathematical Research, 2018. 34(04): 289-295.

[10] Zhang Jian-Wen, Wang Dan-Xia,Wu Run-Heng. Global solutions for a kind of generalized Sine-Gordon equation with strong damping[J]. Acta Physica Sinica, 2008. 57(4) :2021-2025.

[11] LI Huan,LIN Guo-guang. The long time behavior of a damped Sine-Gordon equation[J]. Journal of Yunnan University, 2010. 32(S1): 305-309.

[12] Zhou Shengfan. Dimension of the Globa Attractor for the Damped Sine-Gordon Equation[J]. Acta mathematica sinica, 1996. 39(5): 597-601.

[13] YANG Shuang,Wang REN-hai, LI Yang-rong,SHE Lian-bing. Pullback Dynamics for Non-Autonomous Stochastic Sine-Gordon Equations[J]. Journal of Southwest University(Natural Science Edition), 2018. 40(02): 70-77.

[14] Fu Qi YIN, Lin Fang LIU,Cui Hui XIAO. PeriodicD-Pullback Attractor for Stochastic Sine-Gordon Equation on Unbounded Domain[J]. Acta mathematica sinica(chinese series), 2014(6): 1127-1140.

[15] TANG Qiao-lan, LI Fu-zhi,LI Yang-rong. D-Pullback Attractors for a class of Non-Autonomous Stochastic Damped Sine-Gordon Equations[J].Journal of Southwest University(Natural Science Edition), 2018. 40(06): 101-108.

[16] Ran Y and Shi Q, Pullback attractors for a nonautonomous damped wave equation with infinite delays in weighted space[J]. Mathematical Methods in the Applied Sciences, 2018. 41(12).

[17] Freitas M M, Pullback attractors for non‐autonomous porous elastic system with nonlinear damping and sources terms[J]. Mathematical Methods in the Applied Sciences, 2020. 43(2).

[18] Park J Y and Kang J R, Pullback -attractors for non-autonomous suspension bridge equations[J]. Nonlinear Analysis, 2009. 71(10): 4618-4623.

[19] Caraballo T, Łukaszewicz G and Real J, Pullback attractors for asymptotically compact non-autonomous dynamical systems[J]. Nonlinear Analysis, 2005. 64(3):484-498.

[20] Sun C, Cao D and Duan J, Non-autonomous dynamics of wave equations with nonlinear damping and critical nonlinearity[J]. Mathematics, 2006(7).

[21] Wang B, Periodic Random Attractors for Stochastic Navier-Stokes Equations on Unbounded Domains[J]. Electronic Journal of Differential Equations, 2012. 2012(59).